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Connections between MV_n algebras and n-valued Lukasiewicz–Moisil algebras — II

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Abstract

I introduce in the MV_n algebras of Revaz Grigolia two chains of unary operations, which are the key in establishing many connections between these algebras and n-valued Lukasiewicz–Moisil algebras (LM_n algebras for short). The study has three parts. It is self-contained as much as possible. The main result of the first part is that MV_4 algebras coincide with LM_4 algebras. The larger class of 'relaxed'-MV_n algebras is also introduced and studied. This class is related to the class of generalized LM_n pre-algebras. The main results of the second part are that, for $n \geqslant 5$, any MV_n algebra is an LM_n algebra and that the canonical MV_n algebra can be identified with the canonical LM_n algebra. In the third part, the class of good LM_n algebras and the class of $\oplus$-proper LM_n algebras are introduced and studied. $\oplus$-proper LM_n algebras coincide (can be identified) with Cignoli's proper n-valued Lukasiewicz algebras. MV_n algebras coincide with good LM_n algebras and with $\oplus$-proper LM_n algebras ($n \geqslant 2$). I also give the construction of an LM_3 (LM_4) algebra from the odd (respectively even)-valued LM_n algebra ($n \geqslant 5$), which proves that LM_4 algebras are as much important as LM_3 algebras; the MV_n algebras help us to see that. © 1999 Elsevier Science B.V. All rights reserved

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5. The case $n \geqslant 5$: Given an MV_n algebra

We have seen in the first part of the study [11] that for $n \in \{3,4\}$ any MV_n algebra is an LM_n algebra. We shall see in this section that this is also true for every $n \geqslant 5$.

Lemma 5.1. *Let A be a relaxed-MV_n algebra. Then we have*
(i) $s'_{n-1}(x^2) = s'_{n-2}x$ and
(i′) $s_1(2x) = s_2 x$.

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Proof. Immediate, by [11, 3.3].

Corollary 5.2. *Let $\mathcal{A}$ be a relaxed-*MV_n *algebra and* $n = 2k(k \geqslant 2)$. *Then the axioms* $(M13)$ *and* $(M13')$ *are both equivalent to* $s'_{n-2} = s_2$.

Proof. If n is an even number, then $n-1$ is an odd number. Hence $n-1$ is not divided by 2 and for $j=2$, $(M13)$ and $(M13')$ give, by [11, 3.6] for $j=2$:
$(M13) \iff s_2 x \leqslant s'_{n-1}(x \cdot x)$ and $(M13') \iff s_1(x \oplus x) \leqslant s'_{n-2}x$. Then, by Lemma 5.1, we get:
$(M13) \iff s_2 x \leqslant s'_{n-2}x$ and $(M13') \iff s_2 x \leqslant s'_{n-2}x$. But, by [11, 3.4(iv)] we also have $s'_{n-2}x \leqslant s_2 x$. Hence:
$(M13) \iff (M13') \iff s'_{n-2} = s_2$. $\square$

Proposition 5.3. *Let $\mathcal{A}$ be a relaxed-*MV_n *algebra* $(n \geqslant 3)$. *Then the following hold*:
(i) $s_1 0 = 0, \quad s_1 1 = 1, \quad s_{n-1} 0 = 0, \quad s_{n-1} 1 = 1$;
(ii) $s'_{n-1}(x \cdot [(j-1)x]) \leqslant s_j x$ *and*
(ii′) $s'_{n-j}x \leqslant s_1(x \oplus x^{j-1})$, *for every* j, $1 \leqslant j \leqslant n-1$.

Proof. $s_1 0 = 0^{n-1} = 0$, by [11, (M4′)], [11, 1.3].
$s_1 1 = 1^{n-1} = 1$, by [11, (M5′)], [11, 1.3].
$s_{n-1} 0 = (n-1)0 = 0$, by [11, 3.4, (M5), 1.3].
$s_{n-1} 1 = (n-1)1 = 1$, by [11, 3.4, (M4), 1.3]. Thus (i) holds.
To prove (ii), let $x \in A$, $j \in J$. Then let $T = (s'_{n-1}(x \cdot [(j-1)x]))^- \vee s_j x$. Then $T = s_1((x \cdot [(j-1)x])^-) \vee s_j x = (x^- \oplus ((j-1)x)^-)^{n-1} \vee (jx)^{n-1} = (x^- \oplus ((j-1)x)^-)^{n-1} \vee (x \oplus (j-1)x)^{n-1}$, by [11, 3.4(iii), (M6′), 1.3]. Let $Y = ((j-1)x)^-$. Then $T = (x^- \oplus Y)^{n-1} \vee (x \oplus Y^-)^{n-1}$. But $(x \oplus Y^-) \vee (x^- \oplus Y) = 1$, by [11, (M22)]. It follows that $T = 1$, by [11, (M23)] and hence $s'_{n-1}(x \cdot [(j-1)x]) \leqslant s_j x$, by [11, 3.4(i)]. Thus (ii) holds. By duality we get (ii′). $\square$

Corollary 5.4. *Let $\mathcal{A}$ be an* MV_n *algebra* $(n \geqslant 4)$. *Then*:
(i) $s'_{n-1}(x \cdot [(j-1)x]) = s_j x$ *and*
(i′) $s'_{n-j}x = s_1(x \oplus x^{j-1})$,
for $1 \leqslant j \leqslant n-1$, *$j$ does not divide* $n-1$.

Proof. By Proposition 5.3 and [11, 3.6]. $\square$

Remark 5.5. (1) The axiom $(M13) \iff (x \oplus (j-1)x)^{n-1} \wedge (x^- \oplus [(j-1)x]^-)^{n-1} = 0 \iff (x \oplus Y^-)^{n-1} \wedge (x^- \oplus Y)^{n-1} = 0$, where $Y = [(j-1)x]^-$, with $j \in J$, $1 \leqslant j \leqslant n-1$, j does not divide $n-1$, by [11, 1.10, 3.2].

(2) We obtain another proof of Corollary 5.2 by using Corollary 5.4 (for $j=2$) and Lemma 5.1.

Lemma 5.6. *Let* $\mathscr{L}_n$ *be the canonical* MV_n *algebra* $(n \geqslant 5)$.

- *If $n = 2k$ ($k \geqslant 3$), then the following hold*:
  (i) $x \cdot [(j-1)x] = x \cdot (jx)$ *and*
  (i′) $x \oplus x^{j-1} = x \oplus x^j$, *for every j with $k+1 \leqslant j \leqslant n-2 = 2k-2$,*
  i.e. $x \cdot (kx) = x \cdot [(k+1)x] = \cdots = x \cdot [(2k-2)x]$ *and*
  $x \oplus x^{2k-2} = \cdots = x \oplus x^{k+1} = x \oplus x^k$, *respectively.*
- *If $n = 2k+1$ ($k \geqslant 2$), then the following hold*:
  (ii) $x \cdot [(j-1)x] = x \cdot (jx)$ *and* (ii′) $x \oplus x^{j-1} = x \oplus x^j$,
  for every j with $k+1 \leqslant j \leqslant n-2 = 2k-1$,
  i.e. $x \cdot (kx) = x \cdot [(k+1)x] = \cdots = x \cdot [(2k-1)x]$ *and*
  $x \oplus x^{2k-1} = \cdots = x \oplus x^{k+1} = x \oplus x^k$, *respectively.*

Proof.
- If $n = 2k$ ($k \geqslant 3$), then

$$L_{2k} = \left\{0, \frac{1}{2k-1}, \frac{2}{2k-1}, \ldots, \frac{k-1}{2k-1}, \frac{k}{2k-1}, \ldots, \frac{2k-2}{2k-1}, 1\right\}$$

and

$$kx = \min(1, kx) = \begin{cases} 1, & kx \geqslant 1, \\ kx, & 0 < kx < 1, \\ 0, & kx = 0. \end{cases}$$

(a) If $kx \geqslant 1$, then $kx = 1$ and since $kx \leqslant jx$, for $k \leqslant j \leqslant 2k-2$, it follows that $jx = 1$; hence $x \cdot (jx) = x \cdot 1 = x$, for $k \leqslant j \leqslant 2k-2$.
(b) If $0 < kx < 1$, then let $x = i/(2k-1)$. Then

$$0 < kx < 1 \Longleftrightarrow 0 < \frac{ki}{2k-1} < 1 \Longleftrightarrow 0 \leqslant ki \leqslant 2k-1$$
$$\Longleftrightarrow 0 < i < \frac{2k-1}{k} = 2 - \frac{1}{k} \Longleftrightarrow i = 1 \Longleftrightarrow x = \frac{1}{2k-1}.$$

Then for j such that $k \leqslant j \leqslant 2k-2$ we have $kx \leqslant jx \leqslant (2k-2)x$. Hence $x \cdot (kx) \leqslant x \cdot (jx) \leqslant x \cdot ((2k-2)x)$. But $x \cdot (kx) = \max(0, x + kx - 1) = \max(0, (k+1)/(2k-1) - 1) = 0$ and $x \cdot ((2k-2)x) = \max(0, x + (2k-2)x - 1) = \max(0, (2k-1)/(2k-1) - 1) = 0$ and then $0 \leqslant x \cdot (jx) \leqslant 0$. Hence $x \cdot (jx) = 0$, for $k \leqslant j \leqslant 2k-2$.
(c) If $kx = 0$, then $x = 0$. Hence $jx = 0$, for $k \leqslant j \leqslant 2k-2$ and hence $x \cdot (jx) = 0 \cdot 0 = 0$, for $k \leqslant j \leqslant 2k-2$.

Thus (i) holds. (i′) follows by duality.
- If $n = 2k+1$ ($k \geqslant 2$), then

$$L_{2k+1} = \left\{0, \frac{1}{2k}, \frac{2}{2k}, \ldots, \frac{k-1}{2k}, c = \frac{k}{2k}, \frac{k+1}{2k}, \ldots, \frac{2k-1}{2k}, 1\right\}$$

and the proof is similar. □

Remark 5.7. (i) In any MV_n algebra ($n \geqslant 5$), the chains of elements:

$$(x \cdot (jx))_{j \in J} \quad \text{and} \quad (x \oplus x^j)_{j \in J}$$

where $J = \{1, 2, 3, \ldots, n-1\}$, are 'contracted' to have each at most $k+1$ distinct elements, namely:

- if $n = 2k$ ($k \geqslant 3$),
  $x \cdot x \leqslant x \cdot (2x) \leqslant x \cdot (3x) \leqslant \cdots \leqslant x \cdot (kx) = \cdots = x \cdot ((2k-2)x) \leqslant x$ and
  $x \leqslant x \oplus x^{2k-2} = \cdots = x \oplus x^k \leqslant \cdots \leqslant x \oplus x^3 \leqslant x \oplus x^2 \leqslant x \oplus x$.
- if $n = 2k+1$ ($k \geqslant 2$),
  $x^2 \leqslant x \cdot (2x) \leqslant x \cdot (3x) \leqslant \cdots \leq x \cdot (kx) = \cdots = x \cdot ((2k-1)x) \leqslant x$ and
  $x \leqslant x \oplus x^{2k-1} = \cdots = x \oplus x^k \leqslant \cdots \leqslant x \oplus x^3 \leqslant x \oplus x^2 \leqslant 2x$,
  by [11, 3.13], Lemma 5.6 and [11, 1.12].

(ii) In any MV_n algebra ($n \geqslant 5$),

- if $n = 2k$ ($k \geqslant 3$), then every j verifying $k+1 \leqslant j \leqslant 2k-2$ does not divide $n-1$, since

$$J = \{1, 2, \ldots, k-1, c = k, k+1, \ldots, 2k-2, 2k-1\}$$

  and $j > c$, c being the center of J;
- if $n = 2k+1$ ($k \geqslant 2$), then every j verifying: $k+1 \leqslant j \leqslant 2k-1$, does not divide $n-1$, since it is too great:

$$J = \{1, 2, 3, \ldots, k, k+1, \ldots, 2k\}.$$

Lemma 5.8. *Let $\mathscr{L}_n$ be the canonical* MV_n *algebra. If $n \geqslant 5$ ($n = 2k$ or $n = 2k+1$), then*

$$s'_{n-1}(x \cdot (kx)) = s_k x.$$

Proof.

$$kx = \min(1, kx) = \begin{cases} 1, & kx \geqslant 1, \\ kx, & 0 < kx < 1, \\ 0, & kx = 0. \end{cases}$$

- If $n = 2k$, $k \geqslant 3$, then:
  (a) If $kx \geqslant 1$, then $kx = 1$ and $x \cdot (kx) = x \neq 0$, by [11, (M5′), (M5)]. Hence, $s'_{n-1}(x \cdot (kx)) = (n-1)(x \cdot (kx)) = (n-1)x = 1$, by [11, 1.13, 3.2] and $s_k x = (kx)^{n-1} = 1^{n-1} = 1$, by [11, (M5′)].
  (b) If $0 < kx < 1$, then $x = 1/(2k-1)$ and $x \cdot (kx) = 0$ as was seen in the proof of Lemma 5.6. Hence $s'_{n-1}(x \cdot (kx)) = (n-1)0 = 0$ and $s_k x = (kx)^{n-1} = (k/(2k-1))^{n-1} = 0$, by [11, 1.13].
  (c) If $kx = 0$, then $x = 0$. Hence $kx = 0$ and $x \cdot (kx) = 0 \cdot 0 = 0$. It follows that $s'_{n-1}(x \cdot (kx)) = 0 = s_k x$.
- The proof in the case $n = 2k+1$, $k \geqslant 2$ is similar. □

Proposition 5.9. *Let $\mathscr{L}_n = (L_n, \oplus, \cdot, ^-, 0, 1)$ be the canonical* MV_n *algebra* ($n \geqslant 5$). *Then the canonical* $g.\mathrm{LM}_n$ *pre-algebra,*

$$\mathscr{L}_n = (L_n, \vee, \wedge, ^-, (s_j)_{j\in J}, (s'_j)_{j\in J}, 0, 1),$$

verifies the additional properties:
- *If* $n = 2k$ ($k \geqslant 3$), *then*
  (0) $s'_{n-2} = s_2$,
  (i) $s_{j-1} = s_j$ *and*
  (i′) $s'_{n-j+1} = s'_{n-j}$,
  for $k+1 \leqslant j \leqslant n-2 = 2k-2$, *i.e.* $s_k = s_{k+1} = \cdots = s_{2k-2}$ *and* $s'_2 = s'_3 = \cdots = s'_k$, *respectively.*
- *If* $n = 2k+1$ ($k \geqslant 2$), *then* (i) $s_{j-1} = s_j$ *and*
  (i′) $s'_{n-j+1} = s'_{n-j}$,
  for $k+1 \leqslant j \leqslant n-2 = 2k-1$, *i.e.* $s_k = s_{k+1} = \cdots = s_{2k-1}$ *and* $s'_2 = s'_3 = \cdots = s'_{k+1}$, *respectively.*

Proof.
- If $n = 2k$, then (0) is Corollary 5.2. To prove (i), we have, cf. Lemma 5.6, that

$$x \cdot (kx) = x \cdot [(k+1)x] = \cdots = x \cdot [(2k-3)x] = x \cdot [(2k-2)x];$$

then $s'_{n-1}(x \cdot (kx)) = s_k x$, by Lemma 5.8 and $s'_{n-1}(x \cdot (kx)) = s_{k+1}x$,

$$s'_{n-1}(x \cdot [(k+1)x]) = s_{k+2}x, \ldots, s'_{n-1}(x \cdot [(2k-3)x]) = s_{2k-2}x,$$

by Corollary 5.4, Remarks 5.7(ii) ($j = 2k-1 = n-1$ divides $n-1$). Thus (i) holds. The equalities (i′) follow by duality.
- If $n = 2k+1$, we have a similar proof. □

Remark 5.10. (i) In the canonical MV_n algebra $\mathscr{L}_n$, $n = 2k$ ($k \geqslant 3$), the chains of operations $(s_j)_{j\in J}$ and $(s'_j)_{j\in J}$, where

$$J = \{1,\ 2,\ \ldots\ ,\ k-1,\ k,\ k+1,\ \ldots\ ,\ 2k-2,\ 2k-1\},$$

are 'contracted' as follows:

$$s_1 \leqslant s_2 \leqslant \cdots \leqslant \underbrace{s_k = s_{k+1} = \cdots = s_{2k-2}} \leqslant s_{2k-1} \quad \text{and}$$

$$s'_1 \leqslant \underbrace{s'_2 = s'_3 = \cdots = s'_k} \leqslant s'_{k+1} \leqslant \cdots \leqslant s'_{2k-3} \leqslant s'_{2k-2} \leqslant s'_{2k-1}.$$

Each chain has at most $k+1$ distinct operations. Consequently, the canonical $g.\mathrm{LM}_n$ pre-algebra $\mathscr{L}_n$ has, in both chains together, at most $2(k+1)-3 = 2k-1 = n-1$ distinct operations, since $s_1 = s'_1$, and $s_{n-1} = s'_{n-1}$, by [11, 3.10] and $s_2 = s'_{n-2}$, by Proposition 5.9(0). We shall see further (Corollaries 5.12) that it has exactly $n-1$ distinct operations only for $n = 6$ and $n = 8$. For example, the canonical pre-algebra $\mathscr{L}_{10}$ has 7 distinct operations: $s'_1 = s_1$ $(=r_1)$, $s'_2 = \cdots = s'_5$ $(=r_2)$, $s'_6 = s'_7$ $(=r_3)$,

$s'_8 = s_2\ (=r_5)$, $s_3 = s_4\ (=r_7)$, $s_5 = \cdots = s_8\ (=r_8)$, $s'_9 = s_9\ (=r_9)$ (notice that r_4 and r_6 are missing) and the canonical pre-algebra $\mathscr{L}_{20}$ has only 13 distinct operations: $s'_1 = s_1\ (=r_1)$, $s'_2 = \cdots = s'_{10}\ (=r_2)$, $s'_{11} = s'_{12} = s'_{13}\ (=r_3)$, $s'_{14} = s'_{15}\ (=r_4)$, $s'_{16}\ (=r_5)$, $s'_{17}\ (=r_7)$, $s'_{18} = s_2\ (=r_{10})$, $s_3\ (=r_{13})$, $s_4\ (=r_{15})$, $s_5 = s_6\ (=r_{16})$, $s_7 = s_8 = s_9\ (=r_{17})$, $s_{10} = \ldots = s_{18}\ (=r_{18})$, $s'_{19} = s_{19}\ (=r_{19})$ (notice that r_6, r_8, r_9, r_{11}, r_{12}, r_{14} are missing), where $(r_j)_{j\in J}$ are the canonical operations from the canonical LM_n algebra $\mathscr{L}_n$.

(i′) In the canonical MV_n algebra $\mathscr{L}_n$, $n = 2k+1$ $(k\geqslant 2)$, the chains $(s_j)_{j\in J}$ and $(s'_j)_{j\in J}$, where

$$J = \{1, \ldots, k, k+1, \ldots, 2k\},$$

are ‘contracted’ as follows:

$$s_1 \leqslant s_2 \leqslant \cdots \leqslant \underbrace{s_k = s_{k+1} = \cdots = s_{2k-1}} \leqslant s_{2k} \quad \text{and}$$

$$s'_1 \leqslant \underbrace{s'_2 = s'_3 = \cdots = s'_k = s'_{k+1}} \leqslant \cdots \leqslant s'_{2k-1} \leqslant s'_{2k}.$$

Each chain has at most $k+1$ distinct operations. Consequently,the canonical $g.LM_n$ pre-algebra $\mathscr{L}_n$ has, in both chains together, at most $2(k+1)-2 = 2k = n-1$ distinct operations, since $s_1 = s'_1$ and $s_{n-1} = s'_{n-1}$, by [11,3.10]. We shall see further (Corollaries 5.12) that it has exactly $n-1$ distinct operations only for $n \in \{5, 7, 9, 11\}$. For example, the canonical pre-algebra $\mathscr{L}_{13}$ has 10 distinct operations: $s'_1 = s_1\ (=r_1)$, $s'_2 = \cdots = s'_7\ (=r_2)$, $s'_8 = s'_9\ (=r_3)$, $s'_{10}\ (=r_4)$, $s'_{11}\ (=r_6)$, $s_2\ (=r_7)$, $s_3\ (=r_9)$, $s_4 = s_5\ (=r_{10})$, $s_6 = \cdots = s_{11}\ (=r_{11})$, $s_{12} = s'_{12}\ (=r_{12})$ (notice that r_5 and r_8 are missing).

By Proposition 5.9 we get the following

Proposition 5.11. *Let* $(L_n, \oplus, \cdot, ^-, 0, 1)$ *be the canonical* MV_n *algebra and let* $(r_j)_{j\in J}$ *be the canonical* LM_n *operations.*

- *If* $n = 2k$ $(k\geqslant 3)$, *then*

(i)

$$r_1 = s'_1\ (= s_1),$$

$$r_2 = s'_2\ (= \cdots = s'_k),$$

$$r_3 = s'_{k+1},$$

$$r_k = s_2\ (= s'_{2k-2}),$$

$$r_{2k-3} = s_{k-1},$$

$$r_{2k-2} = s_k\ (= \cdots = s_{2k-2}),$$

$$r_{2k-1} = s_{2k-1}\ (= s'_{2k-1}),$$

(ii) *and if* $k\geqslant 5$, *then* $s_{k-2} = s_{k-1}$.

- *If $n = 2k + 1$ $(k \geqslant 2)$, then*
  (i′)

$$\begin{aligned} r_1 &= s'_1\ (= s_1),\\ r_2 &= s'_2\ (= \cdots = s'_{k+1}),\\ r_3 &= s'_{k+2},\\ r_k &= s'_{2k-1},\\ r_{k+1} &= s_2,\\ r_{2k-2} &= s_{k-1},\\ r_{2k-1} &= s_k\ (= \cdots = s_{2k-1}),\\ r_{2k} &= s_{2k}\ (= s'_{2k}), \end{aligned}$$

(ii′) *and if $k \geqslant 6$, then $s_{k-2} = s_{k-1}$.*

Proof.
- $n = 2k$ $(k \geqslant 3)$. Then

$$L_{2k} = \left\{0, \frac{1}{2k-1}, \ldots, \frac{k-1}{2k-1}, \frac{k}{2k-1}, \frac{2k-2}{2k-1}, 1\right\}$$

To prove (i), let us put $x = i/(2k-1)$, $i = \overline{0, 2k-1}$. By [11, (L4), 3.4(iii)], it is sufficient to prove the last four relations. Indeed, then we have

$$\begin{aligned} r_1x^- &= (r_{2k-1}x)^- = (s'_{2k-1}x)^- = s_1x^-,\\ r_2x^- &= (r_{2k-2}x)^- = (s_kx)^- = s'_{2k-k}x^- = s'_kx^-,\\ r_3x^- &= (r_{2k-3}x)^- = (s_{k-1}x)^- = s'_{2k-(k-1)}x^- = s'_{k+1}x^-. \end{aligned}$$

— On the one hand, we have that

$$r_k\left(\frac{i}{2k-1}\right) = \begin{cases} 0, & k+i < 2k\\ 1, & k+i \geqslant 2k \end{cases} = \begin{cases} 0, & i = \overline{0, k-1},\\ 1, & i = \overline{k, 2k-1}. \end{cases}$$

On the other hand, we have that

$$2x = \frac{i}{2k-1} \oplus \frac{i}{2k-1} = \min\left(1, \frac{2i}{2k-1}\right) = \begin{cases} \dfrac{2i}{2k-1} \neq 1, & i = \overline{0, k-1},\\ 1, & i = \overline{k, 2k-1}. \end{cases}$$

Hence

$$s_2x = (2x)^{n-1} = \begin{cases} 0, & i = \overline{0, k-1}\\ 1, & i = \overline{k, 2k-1} \end{cases} \quad \text{by [11, 1.13].}$$

Thus $r_kx = s_2x$.

— On the one hand, we have that

$$r_{2k-1}\left(\frac{i}{2k-1}\right)=\begin{cases}0, & 2k-1+i<2k\\ 1, & 2k-1+i\geqslant 2k\end{cases}=\begin{cases}0, & i=0,\\ 1, & i=\overline{1,2k-1}.\end{cases}$$

On the other hand, we have that $s_{2k-1}(i/(2k-1))=((2k-1)i/(2k-1))^{n-1}$; but

$$(2k-1)\frac{i}{2k-1}=\min\left(1,\frac{(2k-1)i}{2k-1}\right)=\begin{cases}0\neq 1, & i=0,\\ 1, & i=\overline{1,2k-1}.\end{cases}$$

Hence

$$s_{2k-1}\left(\frac{i}{2k-1}\right)=\begin{cases}0, & i=0\\ 1, & i=\overline{1,2k-1}\end{cases}\quad\text{by [11, 1.13].}$$

Thus $r_{2k-1}x=s_{2k-1}x$.

— On the one hand, we have that

$$r_{2k-2}\left(\frac{i}{2k-1}\right)=\begin{cases}0, & 2k-2+i<2k\\ 1, & 2k-2+i\geqslant 2k\end{cases}=\begin{cases}0, & i=0,1,\\ 1, & i=\overline{2,2k-1}.\end{cases}$$

On the other hand, we have that

$$s_k\left(\frac{i}{2k-1}\right)=\left(k\frac{i}{2k-1}\right)^{n-1};$$

but

$$k\frac{i}{2k-1}=\min\left(1,\frac{ki}{2k-1}\right)=\begin{cases}\dfrac{ki}{2k-1}, & ki<2k-1\\ 1, & ki\geqslant 2k-1\end{cases}$$
$$=\begin{cases}\dfrac{ki}{2k-1}\neq 1, & i=0,1,\\ 1, & i=\overline{2,2k-1}.\end{cases}$$

Hence

$$s_k\left(\frac{i}{2k-1}\right)=\begin{cases}0, & i=0,1\\ 1, & i=\overline{2,2k-1}\end{cases}\quad\text{by [11, 1.13].}$$

Thus $r_{2k-2}x=s_kx$.

— On the one hand, we have that

$$r_{2k-3}\left(\frac{i}{2k-1}\right)=\begin{cases}0, & 2k-3+i<2k\\ 1, & 2k-3+i\geqslant 2k\end{cases}=\begin{cases}0, & i<3,\\ 1, & i\geqslant 3.\end{cases}$$

On the other hand, we have that

$$s_{k-1}x=((k-1)x)^{n-1};$$

but

$$(k-1)x = \min(1,(k-1)x)$$

$$= \begin{cases} \dfrac{(k-1)i}{2k-1}, & (k-1)i < 2k-1 \\ 1, & (k-1)i \geqslant 2k-1 \end{cases} = \begin{cases} \dfrac{(k-1)i}{2k-1} \neq 1, & i = 0,1,2, \\ 1, & i = \overline{3,2k-1}. \end{cases}$$

Hence

$$s_{k-1}x = \begin{cases} 0, & i < 3 \\ 1, & i \geqslant 3 \end{cases} \quad \text{by [11, 1.13]}.$$

Thus $r_{2k-3}x = s_{k-1}x$.
To prove (ii), for $x = i/(2k-1)$, we have: $(k-2)x = \min(1,(k-2)x)$

$$= \begin{cases} \dfrac{(k-2)i}{2k-1}, & (k-2)i < 2k-1 \\ 1, & (k-2)i \geqslant 2k-1 \end{cases} = \begin{cases} \dfrac{(k-2)i}{2k-1} \neq 1, & i = 0,1,2 \\ 1, & i = \overline{3,2k-1} \end{cases},$$

since $(k-2)i < 2k-1$ for $i = 0,1,2$ while $(k-2)i \geqslant 2k-1$ for $i \geqslant 3$ (proof by induction). Then

$$s_{k-2}x = ((k-2)x)^{n-1} = \begin{cases} 0, & i < 3 \\ 1, & i \geqslant 3 \end{cases} = s_{k-1}x.$$

- If $n = 2k+1$ $(k \geqslant 2)$, the proof is similar. □

Corollary 5.12. *Let $\mathscr{L}_n$ be the canonical* MV$_n$ *algebra.*

(i) *If $n = 2k$ $(k \in \{3,4\})$ or if $n = 2k+1$ $(k \in \{2, 3, 4, 5\})$, then the chains $(s_j)_{j \in J}$ and $(s'_j)_{j \in J}$ have together exactly n-1 distinct elements, which coincide with the canonical* LM$_n$ *operations.*

(ii) *If $n = 2k$ $(k \geqslant 5)$ or if $n = 2k+1$ $(k \geqslant 6)$, then the above chains have together less than $n-1$ distinct elements.*

Proof. (i) follows by Proposition 5.11(i) and (i′). Indeed,

- if $n = 2k$: if $k = 3$ $(n = 6)$, we get: $r_1 = s'_1 = s_1$, $r_2 = s'_2 = s'_3$, $r_3 = s'_4 = s_2$, $r_4 = s_3 = s_4$, $r_5 = s_5 = s'_5$; if $k = 4$ $(n = 8)$, we get: $r_1 = s'_1 = s_1$, $r_2 = s'_2 = s'_3 = s'_4$, $r_3 = s'_5$, $r_4 = s'_6 = s_2$, $r_5 = s_3$, $r_6 = s_4 = s_5 = s_6$, $r_7 = s_7 = s'_7$;
- if $n = 2k+1$; if $k = 2$ $(n = 5)$, we get: $r_1 = s'_1 = s_1$, $r_2 = s'_2 = s'_3$, $r_3 = s_2 = s_3$, $r_4 = s_4 = s'_4$; if $k = 3$ $(n = 7)$, we get: $r_1 = s'_1 = s_1$, $r_2 = s'_2 = s'_3 = s'_4$, $r_3 = s'_5$, $r_4 = s_2$, $r_5 = s_3 = s_4 = s_5$, $r_6 = s_6 = s'_6$; if $k = 4$ $(n = 9)$, we get: $r_1 = s'_1 = s_1$, $r_2 = s'_2 = s'_3 = s'_4 = s'_5$, $r_3 = s'_6$, $r_4 = s'_7$, $r_5 = s_2$, $r_6 = s_3$, $r_7 = s_4 = s_5 = s_6 = s_7$, $r_8 = s_8 = s'_8$; if $k = 5$ $(n = 11)$, we get: $r_1 = s'_1 = s_1$, $r_2 = s'_2 = s'_3 = s'_4 = s'_5 = s'_6$, $r_3 = s'_7$, $r_5 = s'_9$, $r_6 = s_2$, $r_8 = s_4$, $r_9 = s_5 = s_6 = s_7 = s_8 = s_9$, $r_{10} = s_{10} = s'_{10}$

and by [11, 3.12, 3.4(i)] it follows that $r_4 = s'_8$ and $r_7 = s_3$. Indeed, to prove that $r_7 = s_3$ for instance, let $x = i/10$. Then, on the one hand, we have that

$$r_7 x = r_7\left(\frac{i}{10}\right) = \begin{cases} 0, & 7+i < 11 \\ 1, & 7+i \geqslant 11 \end{cases} = \begin{cases} 0, & i = \overline{0,3}, \\ 1, & i = \overline{4,10}. \end{cases}$$

On the other hand, we have that

$$3x = \left(\frac{i}{10} \oplus \frac{i}{10}\right) \oplus \frac{i}{10} = \min\left(1, \frac{3i}{10}\right) = \begin{cases} \frac{3i}{10}, & i = \overline{0,3}, \\ 1, & i = \overline{4,10}. \end{cases}$$

Hence

$$s_3 x = (3x)^{10} = \begin{cases} 0, & i = \overline{0,3}, \\ 1, & i = \overline{4,10}, \end{cases}$$

by [11, 1.13]. Thus $r_7 x = s_3 x$.

Finally, since the canonical operations $r_1, r_2, \ldots, r_{n-1}$ are all distinct, we get that (i) is true.

(ii) follows by Remarks 5.10(i) and (i′) and by Proposition 5.11(ii) and (ii′). □

Remark 5.13. (i) We can count how many distinct elements do exist in the two chains. We shall take $L_n = \{0, 1, 2, \cdots, n-1\}$ to avoid the work with fractions; then $x \oplus y = \min(n-1, x+y)$. An easy PASCAL program calculates jx, with $2 \leq j \leqslant n-1$ and $x \in L_n$, where $n = 2k$ $(k \geqslant 3)$ (which allows then to calculate $s_j x$, by [11, 1.13]. Remark that it is sufficient to take $j = \overline{2,k}$ and $x = \overline{0,k}$.

Then an easy PASCAL program determines the other operations s_j and s'_j which coincide and calculates the total number of distinct operations in the two chains (for $n = 2k$ $(k \geqslant 5)$). To obtain the programs for $n = 2k+1$ $(k \geqslant 6)$ one must change '$n = 2k$' to '$n = 2k+1$' in both programs and the test '$k \geqslant 5$' to '$k \geqslant 6$' in the second program.

Programs are available from the author on request.

(ii) Since the canonical MV_n algebra $\mathscr{L}_n$ is a linearly ordered relaxed-MV_n algebra, it follows, by [11, 3.12], that $(L_n, \vee, \wedge, ^{-}, (r'_j)_{j \in J'}, 0, 1)$ is an LM_{2n-3} pre-algebra, with $J' = \{1, 2, \ldots, 2n-4\}$. In this pre-algebra, if we take only the distinct operations r'_j, $j \in J'$ and we rename them by $r_j, j \in J$, then: for $n \in \{5, 6, 7, 8, 9, 11\}$ or for $n \in \{4\}$, we obtain an LM_n algebra ($J = \{1, 2, \ldots, n\}$), the canonical one, by Corollaries 5.12(i) and by [11, 4.10(i)], while for $n = 2k$ $(k \geqslant 5)$ or for $n = 2k+1$ $(k \geqslant 6)$, we obtain an LM_m pre-algebra, with $m < n$ ($J = \{1, 2, \ldots, m\}$), by Corollaries 5.12(ii). This 'contraction' of the chain $(r'_j)_{j \in J'}$ to the chain $(r_j)_{j \in J}$ is the effect of the axioms (M13) and (M13') from the definition of an MV_n algebra [11, 1.5] added to the definition of a relaxed-MV_n algebra [11, 1.14].

Definition 5.14. Let $J = \{1, 2, \dots, n-1\}$, $n \geqslant 5$. Let $\mathcal{A} = (A, \oplus, \cdot, ^-, 0, 1)$ be an MV_n algebra. Define $\Phi^p(\mathcal{A}) = (A, \vee, \wedge, ^-, (r_j)_{j\in J}, 0, 1)$ by

$$x \vee y = x \cdot y^- \oplus y, \qquad x \wedge y = (x^- \vee y^-)^-$$

and

(i) if $n = 2k$ ($k \in \{3,4\}$), then $(r_j)_{j\in J}$ are defined by the relations from Proposition 5.11(i);

(i′) if $n = 2k+1$ ($k \in \{2,3,4,5\}$), then $(r_j)_{j\in J}$ are defined by the relations from Proposition 5.11 (i′) and for $k = 5$, $r_4 = s'_8$ and $r_7 = s_3$.

Then we have the following proposition.

Proposition 5.15. *Let $n \in \{6,8\} \cup \{5,7,9,11\}$. If $\mathcal{L}_n$ is the canonical* MV_n *algebra, then $\Phi^p(\mathcal{L}_n)$ is the canonical* LM_n *algebra.*

Proof. By [11, 3.1] $(A, \vee, \wedge, ^-, 0, 1)$ is a De Morgan algebra. Then follow the proof of Corollaries 5.12(i) to see that the above defined $(r_j)_{j\in J}$ are all of the canonical LM_n operations. □

Theorem 5.16. *Let $n \in \{5, 6, 7, 8, 9, 11\}$. If $\mathcal{A}$ is an* MV_n *algebra, then $\Phi^p(\mathcal{A})$ is an* LM_n *algebra.*

Proof. By Proposition 5.15, [11, 1.12] and the converse of [11, 2.8] which is true since the class of LM_n algebras is equational. □

Remark 5.17. (i) Φ^p from Definition 5.14 extends Φ from [11, 4.1 and 4.9].

(ii) In an arbitrary MV_n algebra, $n = 2k$ ($k \geqslant 3$), we have the representation of the chains $(s_j)_{j\in J}$ and $(s'_j)_{j\in J}$ in Fig. 1 which is an extension of [11, Fig. 3] (see [11, Fig. 1] to understand the effect of the axioms [11, (M13) and (M13′)]).

(ii′) In an arbitrary MV_n algebra, $n = 2k+1$ ($k \geqslant 2$), we have the representation of the chains $(s_j)_{j\in J}$ and $(s'_j)_{j\in J}$ in Fig. 2 which is an extension of [11, Fig. 2] (see [11, Fig. 1] to understand the effect of the axioms [11, (M13) and (M13′)]).

At this point of my research I read [19,21,5], so I can go on by using Suchoń's transformation to obtain an extension of the transformation Φ for any $n \geqslant 3$.
Suchoń defines Moisil operators $(\sigma_j)_{j\in J}$ ($\sigma_j = r_{n-j}$) of the canonical LM_n algebra ($n \geqslant 3$) starting from the Lukasiewiczian implication $\to$ and from the negation $-$. He puts

$$B_3(x) = (x^-) \to x \text{ and } B_{j+1}(x) = (x^-) \to B_j(x),\ j \geqslant 3. \tag{1}$$

Then he defines

$$\sigma_1 x = B_n(x), \tag{2}$$

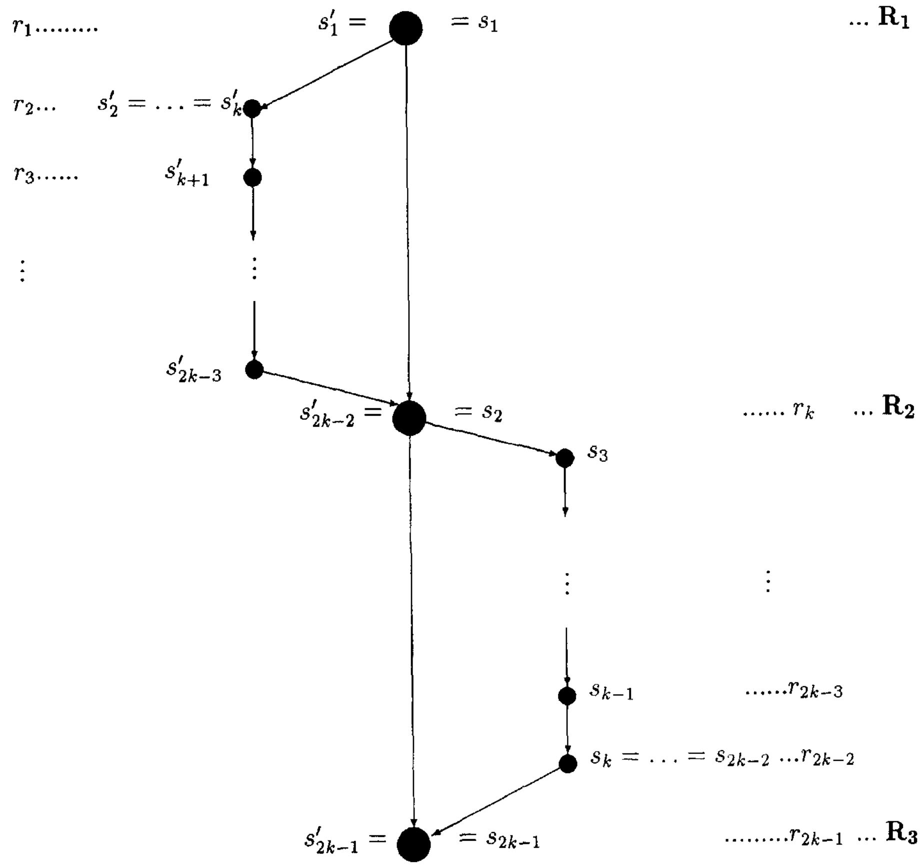

Fig. 1.

$$\text{and for } 1 < j \leqslant [n/2], \qquad \sigma_j x = \begin{cases} \sigma_{n-1}(B_{l+1}(x)), & lj \geqslant n-1, \\ \sigma_{lj}(B_{l+1}(x)), & lj < n-1, \end{cases} \tag{3}$$

where $l = \max\{m \mid m(j-1) < n-1\}$,

$$\text{while } \sigma_{n-j}(x) = (\sigma_j(x^-))^-, \text{ for } \quad 1 \leqslant j \leqslant [n/2]. \tag{4}$$

Suchoń's Moisil operators verify: $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{n-1}$.

Remark 5.18. If we want to use Suchoń's construction, it is convenient to consider not the MV algebra, $(A, \oplus, \cdot, {}^-, 0, 1)$ (or equivalently $(A, \oplus, {}^-, 0)$, by [6]), but the Wajsberg algebra (W algebra for short), $(A, \rightarrow, {}^-, 1)$, defined in [8]. W algebras can be identified with MV algebras, by [8, Theorems 4 and 5]. Namely,

- if $\mathcal{A} = (A, \rightarrow, {}^-, 1)$ is a W algebra and if we define

$$\alpha(\mathcal{A}) = (A, \oplus, \cdot, {}^-, 0, 1)$$

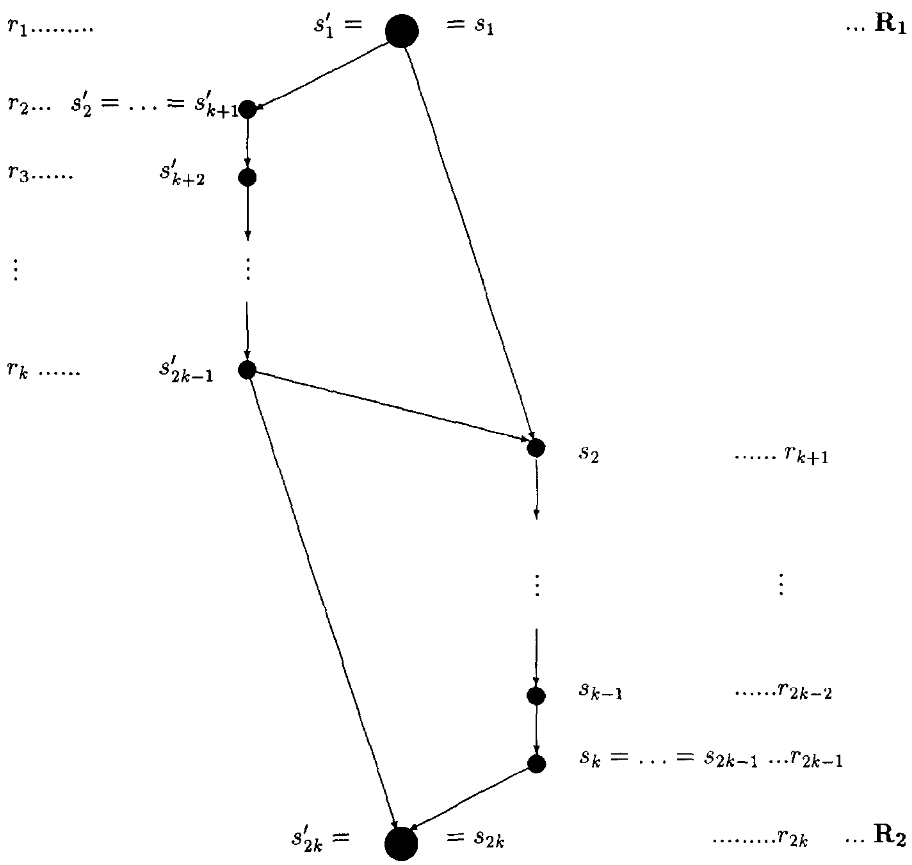

Fig. 2.

by

$$x \oplus y = x^- \to y, \quad x \cdot y = (x^- \oplus y^-)^-, \tag{5}$$

then $\alpha(\mathcal{A})$ is an MV algebra.

- Conversely, if $\mathcal{A} = (A, \oplus, \cdot, ^-, 0, 1)$ is an MV algebra and if we define

$$\beta(\mathcal{A}) = (A, \to, ^-, 1)$$

by

$$x \to y = x^- \oplus y \tag{6}$$

then $\beta(\mathcal{A})$ is a W algebra.

- The maps α, β are mutually inverse.

It follows immediately by (1) that

$$B_3(x) = x \oplus x = 2x \quad \text{and} \quad B_{j+1}(x) = x \oplus B_j(x) = jx, \ j \geqslant 3. \tag{7}$$

By using Suchoń's construction we give the following

Definition 5.19. Let $\mathcal{A} = (A, \oplus, \cdot, ^-, 0, 1)$ be an MV_n algebra $(n \geqslant 3)$. Define

$$\Phi^S(\mathcal{A}) = (A, \vee, \wedge, ^-, (r_j)_{j \in J}, 0, 1)$$

by

$$x \vee y = x \cdot y^- \oplus y, \qquad x \wedge y = (x^- \vee y^-)^-,$$

$$r_{n-1}x = (n-1)x \; (= s_{n-1}x = s'_{n-1}x), \tag{8}$$

$$r_{n-j}x = \begin{cases} r_1(lx) \; (= s_1(lx) = s_l(x)), & lj \geqslant n-1, \\ r_{n-lj}(lx), & lj < n-1, \end{cases} \tag{9}$$

for $1 < j \leqslant [n/2]$,

$$l = \max\{m \mid m(j-1) < n-1\},$$

$$r_j x = (r_{n-j}(x^-))^-, \qquad 1 \leqslant j \leqslant [n/2]. \tag{10}$$

Remark 5.20. $s_1(lx) = (lx)^{n-1} = s_l(x)$ by [11, 3.3].

Then we have the following

Proposition 5.21. *If $\mathcal{L}_n$ is the canonical MV_n algebra $(n \geqslant 3)$, then $\Phi^S(\mathcal{L}_n)$ is the canonical LM_n algebra.*

Proof. In view of [11, 3.1], it is sufficient to prove that $(r_j)_{j \in J}$ are the canonical operations. By (8), (7) and (2) we have $r_{n-1}x = (n-1)x = B_n(x) = \sigma_1 x$. Then, for $1 < j \leqslant [n/2]$, if $lj \geqslant n-1$, then $r_1(lx) = [r_{n-1}((lx)^-)]^- = [\sigma_1((lx)^-)]^- = \sigma_{n-1}(lx) = \sigma_{n-1}(B_{l+1}(x))$ and if $lj < n-1$, then $r_{n-lj}(lx) = \sigma_{lj}(lx) = \sigma_{lj}(B_{l+1}(x))$; hence, $r_{n-j}x = \sigma_j x$. We have $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_{n-1}$ since $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{n-1}$. Then, by [21], $(r_j)_{j \in J}$ are the canonical operations. $\square$

Remark 5.22. We can give a direct proof of Proposition 5.21. Let $(r_j)_{j \in J}$ be the canonical LM_n operations. It is sufficient to prove that the equalities (8) and (9) hold. (8) is true by Proposition 5.11(i) and (i'). To prove (9), let $x = i/(n-1) \in L_n$. Then

$$r_{n-j}x = r_{n-j}\left(\frac{i}{n-1}\right) = \begin{cases} 0, & n-j+i < n \\ 1, & n-j+i \geqslant n \end{cases} = \begin{cases} 0, & i < j, \\ 1, & i \geqslant j \end{cases} \tag{11}$$

and

$$s_l x = (lx)^{n-1} = \begin{cases} 0, & lx \neq 1 \\ 1, & lx = 1 \end{cases} = \begin{cases} 0, & l\dfrac{i}{n-1} \neq 1 \\ 1, & l\dfrac{i}{n-1} = 1, \end{cases} \quad \text{by [10, 1.13].}$$

- If $lj \geqslant n-1$, then $r_{n-j}x = s_l x$, i.e. $i < j \Leftrightarrow li/(n-1) \neq 1$.

Indeed, $i < j \Rightarrow i \leqslant j-1 \Rightarrow li \leqslant l(j-1) < n-1$, by the definition of l. Hence $li/(n-1) < 1$ and thus $li/(n-1) = \min(1, li/(n-1)) = li/(n-1) \neq 1$. If now $li/(n-1) < 1$ and we suppose that $i \geqslant j$, then $li \geqslant lj \geqslant n-1$, hence $li \geqslant n-1$, i.e. $li/(n-1) \geqslant 1$. Hence $li/(n-1) = \min(1, li/(n-1)) = 1$, which contradicts $li/(n-1) < 1$. Thus $i < j$.

- I shall prove now that if $lj < n-1$, then $r_{n-j}x = r_{n-lj}(lx)$.

Indeed, first,

$$lx = l\frac{i}{n-1} = \min\left(1, \frac{li}{n-1}\right) = \begin{cases} \frac{li}{n-1}, & li < n-1, \\ 1, & li \geqslant n-1. \end{cases}$$

— Then, if $li < n-1$, then

$$r_{n-lj}(lx) = r_{n-lj}\left(\frac{li}{n-1}\right) = \begin{cases} 0, & n-lj+li < n \\ 1, & n-lj+li \geqslant n \end{cases}$$
$$= \begin{cases} 0, & li < lj \\ 1, & li \geqslant lj \end{cases} = \begin{cases} 0, & i < j \\ 1, & i \geqslant j \end{cases}, \quad \text{since } l > 0,$$

— and if $li \geqslant n-1$, then

$$r_{n-lj}(lx) = r_{n-lj}(1) = \begin{cases} 0, & n-lj+n-1 < n \\ 1, & n-lj+n-1 \geqslant n \end{cases} = \begin{cases} 0, & n-1 < lj, \\ 1, & n-1 \geqslant lj; \end{cases}$$

but $lj < n-1$, hence $r_{n-lj}(lx) = 1$. Then

$$r_{n-lj}(lx) = \begin{cases} 0, & i < j,\ li < n-1 \\ 1, & i \geqslant j,\ \ li < n-1 \\ 1, & \qquad li \geqslant n-1 \end{cases} = \begin{cases} 0, & i < j \\ 1, & i \geqslant j \end{cases} = r_{n-j}x$$

by ($i < j \Rightarrow li < n-1$) and (11). Thus (9) holds.

Theorem 5.23. *If $\mathcal{A}$ is an* MV_n *algebra* ($n \geqslant 3$), *then $\Phi^S(\mathcal{A})$ is an* LM_n *algebra.*

Proof. By Proposition 5.21 [11, 1.12] and the converse of [11, 2.8]. □

Remark 5.24. (i) Φ from, [11, 4.1 and 4.9] is a particular case of Φ^S from Definition 5.19 (for $n \in \{3,4\}$).

(ii) Φ^p from Definition 5.14 is a particular case of Φ^S from Definition 5.19 (for $n \in \{5,6,7,8,9,11\}$). Indeed,

- if $n = 2k$ ($k \geqslant 3$), then $r_{n-1} = r_{2k-1} = s_{2k-1} = s_{n-1}$ and for $2 \leqslant j \leqslant k$ we get: for $j = 2$, we get $l = n-2$ and $lj \geqslant n-1 \Leftrightarrow 2(2k-2) \geqslant 2k-1 \Leftrightarrow 2k \geqslant 3$; for $j = 3$, we get $l = k-1$ and $lj \geqslant n-1 \Leftrightarrow 3(k-1) \geqslant 2k-1 \Leftrightarrow k \geqslant 2$; for $j = k$, we get $l = 2$ and $lj \geqslant n-1 \Leftrightarrow 2k \geqslant 2k-1$; so in these cases $lj \geqslant n-1$ is always true. Hence $r_{n-j} = s_l$, i.e. $r_{2k-2} = s_{2k-2}$, $r_{2k-3} = s_{k-1}$, $r_{2k-k} = r_k = s_2$. Hence, since if $k = 3$ ($n = 6$), then $j \in \{2, 3 = k\}$ and if $k = 4$ ($n = 8$), then $j \in \{2, 3, 4 = k\}$, it follows that $lj \geqslant n - 1$ for any j.

- If $n = 2k+1$ ($k \geqslant 2$), then $r_{n-1} = r_{2k} = s_{2k} = s_{n-1}$ and for $2 \leqslant j \leqslant k$ we get:for $j = 2$, we get $l = 2k - 1$ and $lj \geqslant n - 1 \Leftrightarrow 2(2k - 1) \geqslant 2k \Leftrightarrow 2k \geqslant 2 \Leftrightarrow k \geqslant 1$; for $j = 3$ ($k \geqslant 3$), we get $l = k - 1$ and $lj \geqslant n - 1 \Leftrightarrow 3(k - 1) \geqslant 2k \Leftrightarrow k \geqslant 3$; for $j = k$, we get $l = 2$ and $lj \geqslant n - 1 \Leftrightarrow 2k \geqslant 2k$; so in these cases $lj \geqslant n - 1$ is always true. Hence, $r_{n-j} = s_l$, i.e. $r_{2k-1} = s_{2k-1}$, $r_{2k-2} = s_{k-1}$, $r_{k+1} = s_2$. Hence, since if $k = 2$ ($n = 5$), then $j \in \{2\}$, if $k = 3$ ($n = 7$), then $j \in \{2, 3 = k\}$ and if $k = 4$ ($n = 9$), then $j \in \{2, 3, 4 = k\}$, it follows that $lj \geqslant n - 1$ for any j; if $k = 5$ ($n = 11$), then $j \in \{2, 3, 4, 5 = k\}$ and for $j = 4$, we get $l = 3$ and $lj = 12 \geqslant 10 = n - 1$.

(iii) In Definition 5.19, if $n = 10$ ($k = 5$), for instance, then $j \in \{2, 3, 4, 5 = k\}$ anf for $j = 4$, we get $l = 2$ and $lj = 8 < 9 = n - 1$; hence, in the canonical algebra $\mathscr{L}_{10}$, $r_{n-j}x = r_6 x = r_{n-lj}(lx) = r_2(2x)$ and consequently $r_4 x = (r_6 x^-)^- = (r_2(x^- \oplus x^-))^- = r_8((x^- \oplus x^-)^-) = r_8(x \cdot x) = r_8(x^2)$, by [11, 2.1(L4)]. For the other canonical operations in $\mathscr{L}_{10}$ see Remark 5.10(i).

If $n = 13$ ($k = 6$), for instance, then $j \in \{2, 3, 4, 5, 6 = k\}$; for $j = 4$, we get $l = 3$ and $l = 12 \geqslant 12 = n - 1$, hence $r_{n-j}x = r_9 x = s_3 x$; but for $j = 5$, we get $l = 2$ and $lj = 10 < 12 = n - 1$ hence, in the canonical algebra $\mathscr{L}_{13}$, we have $r_{n-j}x = r_8 x = r_{n-lj}(lx) = r_3(2x)$ and consequently $r_5 x = (r_8 x^-)^- = (r_3(2x^-))^- = r_{10}((2x^-)^-) = r_{10}((x^- \oplus x^-)^-) = r_{10}(x \cdot x) = r_{10}(x^2)$, by [11, 2.1(L4)]. For the other operations in $\mathscr{L}_{13}$ see Remarks 5.10(i′).

(iv) We obtain [11, 4.3(i), 4.11(i)] as particular cases of Theorem 5.23.

(v) We obtain Theorem 5.16 as a particular case of Theorem 5.23.

(vi) The relation '$lj \geqslant n - 1$' (from Definition 5.19, where $l = \max\{m \mid m(j-1) < n-1\}$) is verified for every $2 \leqslant j \leqslant [n/2]$, $n \geqslant 4$ if and only if $n \in \{4, 5, 6, 7, 8, 9, 11\}$. Indeed, '$\Leftarrow$' follows by remarks (i) and (ii). To prove '$\Rightarrow$', (S. Rudeanu) note first that if $n = 10$ then for $j = 4 < [10/2]$ we have $l = 2$, hence $lj = 8 < 9$. Further, suppose $n \geqslant 12$. Then setting $k = [n/2]$, we have either $n = 2k$ or $n = 2k+1$ and in both cases $k \geqslant 6$, $2(k-2) < 2k-1 \leqslant n-1$ while $3(k-2) \geqslant 2k \geqslant n-1$, hence for $j = k - 1$ we have $l = 2$, therefore $lj = 2k - 2 < n - 1$.

(vii) $\Phi^S = \Phi \Longleftrightarrow n \in \{3, 4\}$ and $\Phi^S = \Phi^p \Longleftrightarrow n \in \{5, 6, 7, 8, 9, 11\}$, by Remark (vi).

(viii) We can write a PASCAL program to determine the canonical operators $(r_j)_{j \in J}$.

(ix) From now on, we shall write simply Φ instead of Φ^S.

Remark 5.25. We have seen that MV_n algebras are particular cases of relaxed-MV_n algebras and relaxed-MV_n algebras are particular cases of MV algebras. We have seen in Remark 5.18 that the analogue of MV algebras are W algebras [8]. We state that the analogue of relaxed-MV_n algebras are the n-bounded W algebras (bounded-W_n algebras for short) defined in [19]. Bounded-W_n algebras are W algebras verifying [19]

$$(E_n) \qquad x \rightarrow_n y = x \rightarrow_{n-1} y,$$

where $x \rightarrow_0 y = y$ and for $0 < n < \omega$, $x \rightarrow_n y = x \rightarrow (x \rightarrow_{n-1} y)$. Our relaxed-$\mathrm{MV}_n$ algebras can be identified with bounded-W_n algebras. Indeed,

$\Rightarrow$ Let$(A, \oplus, \cdot, ^-, 0, 1)$ be a relaxed-MV_n algebra.We put $x \rightarrow y = (x^-) \oplus y$. Then $x \rightarrow_n y = (n(x^-)) \oplus y$, for $n \geqslant 0$. We prove this by induction: $x \rightarrow_0 y = y$; suppose that $x \rightarrow_{n-1} y = (n-1)(x^-) \oplus y$; then $x \rightarrow_n y = x \rightarrow (x \rightarrow_{n-1} y) = x \rightarrow ((n-1)(x^-) \oplus y) = x^- \oplus ((n-1)(x^-) \oplus y) = (x^- \oplus (n-1)(x^-)) \oplus y = n(x^-) \oplus y$. Then $x \rightarrow_n y = n(x^-) \oplus y = (n-1)(x^-) \oplus y = x \rightarrow_{n-1} y$, by [11, (M12)], thus (E_n) holds in the W algebra $(A, \rightarrow, ^-, 1)$.

$\Leftarrow$ Let now $(A, \rightarrow, ^-, 1)$ be a bounded-W_n algebra. We put $x \oplus y = (x^-) \rightarrow y$. Then $nx \oplus y = (x^-) \rightarrow_n y$, for $n \geqslant 0$. We prove this by induction: $0x \oplus y = 0 \oplus y = (0^-) \rightarrow_0 y = y$; suppose that $(n-1)x \oplus y = (x^-) \rightarrow_{n-1} y$; then $nx \oplus y = ((n-1)x \oplus x) \oplus y = x \oplus ((n-1)x \oplus y) = x \oplus ((x^-) \rightarrow_{n-1} y) = (x^-) \rightarrow ((x^-) \rightarrow_{n-1} y) = (x^-) \rightarrow_n y$. Then, by taking y=0, we get $nx = nx \oplus 0 = (x^-) \rightarrow_n 0 = (x^-) \rightarrow_{n-1} 0 = (n-1)x \oplus 0 = (n-1)x$, by (E_n).Thus [11, (M12)] is verified in the MV algebra $(A, \oplus, \cdot, ^-, 0, 1)$ and this completes the proof.

Open problem 5.26. (1) Define the analogue of MV_n algebras, say 'W_n algebras', as particular cases of bounded-W_n algebras. Find the connection between W_n algebras and n valued Wajsberg algebras defined in [19].

(2) Formulate the analogue of Φ and the analogue of Theorem 5.23 for W_n algebras.

6. The case $n \geqslant 5$: Given an LM_n algebra

We have seen in [11] that for $n \in \{3, 4\}$ any LM_n algebra is an MV_n algebra and that MV_n algebras coincide (can be identified) with LM_n algebras. We shall see in this section that, for any $n \geqslant 5$, the canonical MV_n algebra can be identified with the canonical LM_n algebra.

Proposition 6.1. (1) *Given the canonical* LM_n *algebra* $(n \geqslant 3)$

$$\mathscr{L}_n = (L_n, \vee, \wedge, ^-, (r_j)_{j \in J}, 0, 1),$$

define $\Psi(\mathscr{L}_n) = (L_n, \oplus^n, \cdot^n, ^-, 0, 1)$ *by*:

(i) *if* $n = 2k+1$,

$$\begin{aligned} x \oplus^{2k+1} y = {} & (x \vee r_{2k} y) \wedge (y \vee r_{2k} x) \\ & \wedge (x^* \vee r_{2k-1} y) \wedge (y^* \vee r_{2k-1} x) \\ & \vdots \\ & \wedge (x^{(k-1)*} \vee r_{k+1} y) \wedge (y^{(k-1)*} \vee r_{k+1} x), \end{aligned} \tag{12}$$

(i′) *if* $n = 2k$,

$$x \oplus^{2k} y = (x \vee r_{2k-1} y) \wedge (y \vee r_{2k-1} x) \tag{13}$$

$$\wedge (x^* \vee r_{2k-2}y) \wedge (y^* \vee r_{2k-2}x)$$

$$\vdots$$

$$\wedge (x^{(k-1)*} \vee r_k y) \wedge (y^{(k-1)*} \vee r_k x),$$

where x^* is the successor of x and

$$x^{2*} = (x^*)^*, \quad x^{m*} = \left(x^{(m-1)*}\right)^* \tag{14}$$

and

(ii) $$x \cdot^n y = (x^- \oplus^n y^-)^-. \tag{15}$$

Then $\Psi(\mathscr{L}_n)$ is the canonical MV_n *algebra.*
(2) *The maps Φ, from Proposition* 5.21, *and Ψ are mutually inverse.*

Proof.
• To prove (1) note first that the cases $n = 3$ and $n = 4$ are settled by [11, 4.2(ii)], and by [11, 4.10(ii) and 4.13(iv)], respectively. Now take first $n = 2k+1$ ($k \geqslant 2$). In order to simplify the writting, let us consider the canonical LM_n algebra in the equivalent form $(L_{2k+1}, \vee, \wedge, ^-, (r_j)_{j\in J}, 0, 2k)$, where

$$L_{2k+1} = \{0, 1, 2, \ldots, k-1, k, k+1, \ldots, 2k\}$$

and the canonical operations are:

$$x \vee y = \max(x, y), \quad x \wedge y = \min(x, y), \quad x^- = 2k - x,$$

$$r_j x = \begin{cases} 0, & j + x < 2k+1 \\ 2k, & j + x \geqslant 2k+1 \end{cases} \quad x \in L_{1+2k}, j \in J. \tag{16}$$

Then $x^* = \min(2k,\ x+1)$, while the predecessor of x is $^*x = \max(0,\ x-1)$ and if we put $x^{0*} = x$, then

$$x^{m*} = \min(2k, x+m), \quad m \in I\!N. \tag{17}$$

Remark that

$$\text{if } x \leqslant y \text{ then } x^{m*} \leqslant y^{m*} \quad \text{and} \quad \text{if } m \leqslant p \text{ then } x^{m*} \leqslant x^{p*}. \tag{18}$$

Let us note

$$E^{(i)}(x, y) = (x^{i*} \vee r_{2k-i}y) \wedge (y^{i*} \vee r_{2k-i}x), i = \overline{0, k-1}. \tag{19}$$

Then (12) becomes

$$x \oplus^{2k+1} y = \min(E^{(0)}(x, y),\ E^{(1)}(x, y), \ldots, E^{(k-1)}(x, y)). \tag{20}$$

By simplifying the writing, the corresponding equivalent form of the canonical MV_n algebra is $(L_{2k+1}, \oplus, \cdot, ^-, 0, 2k)$, where the canonical operations are

$$x \oplus y = \min(2k, x+y), \quad x \cdot y = \max(0, x+y-2k).$$

Hence, we must prove that the operation $\oplus^{2k+1}$ defined by (20) coincides with the canonical operation $\oplus$. Taking into account the definition of the canonical operation $\oplus$ in the canonical MV_{2k+1} algebra and its commutativity, we see that it sufficient to prove that the operation $\oplus^{2k+1}$ defined by (20) satisfies

(i) $0 \oplus^{2k+1} y = y,\ y \in L_n$;

(ii) $\big(i \oplus^{2k+1} y = \min(2k, i+y),\ y \geqslant i\big),\ i = \overline{1, k-1}$;

(iii) $\big(m \oplus^{2k+1} y = 2k,\ y \geqslant m\big),\ m = \overline{k, 2k}$,

To prove (i)–(iii) we first establish the following relations (we are taking into account the symmetry of $E^{(i)}(x,y)$, $i = \overline{0, k-1}$):

$$E^{(0)}(0,y) = y,\ y \geqslant 0, \tag{21}$$

$$\big(E^{(i)}(i,y) = \min(2k, i+y),\ y \geqslant i\big), \quad i = \overline{1, k-1}, \tag{22}$$

$$\big(\big(E^{(i)}(m,y) = 2k,\ y \geqslant m\big), \quad i < m \leqslant 2k\big), \quad i = \overline{0, k-1}, \tag{23}$$

$$\big(E^{(0)}(0,y) \leqslant E^{(i)}(0,y), y \geqslant 0\big), \quad i = \overline{1, k-1}, \tag{24}$$

$$\big(\big(E^{(i)}(i,y) \leqslant E^{(v)}(i,y),\ y \geqslant i\big),\ v = \overline{i+1, k-1}\big), \quad i = \overline{1, k-2}. \tag{25}$$

Now, in order to prove (21)–(25), we shall prove first the following four helpful relations:

$$r_{2k-i}y = \begin{cases} 0, & y \leqslant i \\ 2k, & y > i \end{cases} \qquad i = \overline{0, k-1}, \tag{26}$$

$$\big(E^{(i)}(i,y) = y^{i*},\ y \geqslant i\big), \quad i = \overline{0, k-1}, \tag{27}$$

$$\left(E^{(i)}(0,y) = \begin{cases} i, & y \leqslant i \\ y^{i*}, & y > i \end{cases} \quad 0 \leqslant y\right), \quad i = \overline{1, k-1}, \tag{28}$$

$$\left(\left(E^{(v)}(i,y) = \begin{cases} i^{v*}, & y \leqslant v \\ y^{v*}, & y > v \end{cases},\ i \leqslant y\right), v = \overline{i+1, k-1}\right), \quad i = \overline{1, k-2}. \tag{29}$$

To prove (26), we have

$$r_{2k-i}y = \begin{cases} 0, & 2k-i+y < 2k+1 \\ 2k, & 2k-i+y \geqslant 2k+1 \end{cases} = \begin{cases} 0, & y < i+1 \\ 2k, & y \geqslant i+1 \end{cases} = \begin{cases} 0, & y \leqslant i, \\ 2k, & y > i. \end{cases}$$

The proof of (27)–(29) is similar, by using (19) and (26).

We can now prove (21)–(25): (21) follows from (27) for $i = 0$; (22) follows from (27), by (17); (23) follows from (19), by (26):

$$\begin{aligned} E^{(i)}(m,y) &= (m^{i*} \vee r_{2k-i}y) \wedge (y^{i*} \vee r_{2k-i}m) \\ &= (m^{i*} \vee 2k) \wedge (y^{i*} \vee 2k) \\ &= 2k \end{aligned}$$

since $y \geqslant m > i$. To prove (24), by using (28) we get if $y \leqslant i$, it follows that $E^{(i)}(0,y) = i \geqslant y$ and if $y > i$, then $E^{(i)}(0,y) = y^{i*} \geqslant y = E^{(0)}(0,y)$, by the definition of y^{i*} and by (21). To prove at last (25) we use (29). Since $1 \leqslant i \leqslant k-2$ and $2 \leqslant i+1 \leqslant v \leqslant k-1$, it follows that $3 \leqslant i+v \leqslant 2k-3 < 2k$. Then $i^{v*} = \min(2k, i+v) = i+v$. If now $y \leqslant v$, then $y+i \leqslant v+i < 2k$ and hence, by (27): $E^{(i)}(i,y) = \min(2k, y+i) = y+i \leqslant v+i = i^{v*} = E^{(v)}(i,y)$. If $y > v$, then $E^{(v)}(i,y) = y^{v*} \geqslant y^{i*} = E^{(i)}(i,y)$, since $v > i$. Thus (25) holds.

Now we are able to prove (i)–(iii). Indeed,

$$0 \oplus^{2k+1} y = \min(E^{(0)}(0,y), E^{(1)}(0,y), \ldots, E^{(k-1)}(0,y)) = E^{(0)}(0,y) = y,$$

by (24) and (21) and thus (i) holds.

For every $i = \overline{1,2k}$, $i \oplus^{2k+1} y = \min(E^{(i)}(i,y), E^{(i+1)}(i,y), \ldots, E^{(k-1)}(i,y))$, by (23). If $1 \leqslant i \leqslant k-1$, then $i \oplus^{2k+1} y = E^{(i)}(i,y) = \min(2k, i+y)$, by (25) and (22) and thus (ii) holds.

If $k \leqslant m \leqslant 2k$, then (23) implies $m \oplus^{2k+1} y = 2k$ and thus (iii) holds. This ends the proof of (i).

The proof of (i′) is similar.

• To prove (2), remark that we have

$$\mathscr{L}_n = \mathscr{L}_n^{(\mathrm{MV}_n)} \longrightarrow \Phi(\mathscr{L}_n) = \mathscr{L}_n^{(\mathrm{LM}_n)} = \mathscr{L}_n \longrightarrow \Psi(\mathscr{L}_n) = \mathscr{L}_n^{(\mathrm{MV}_n)}$$

and that we also have

$$\mathscr{L}_n = \mathscr{L}_n^{(\mathrm{LM}_n)} \longrightarrow \Psi(\mathscr{L}_n) = \mathscr{L}_n^{(\mathrm{MV}_n)} = \mathscr{L}_n \longrightarrow \Phi(\mathscr{L}_n) = \mathscr{L}_n^{(\mathrm{LM}_n)}. \qquad \square$$

Remark 6.2. (i) Ψ from [11, 4.1 and 4.9] is a particular case of Ψ from Proposition 6.1 (for $n \in \{3,4\}$) by (11, 4.13(iv)).

(ii) The transformation Ψ is not polynomial for $n \geqslant 5$, by [11, 4.13(iv)].

(iii) Let $L_n = \{0,1,2,\ldots,n-1\}$. If $n = 2k+1$ $(k \geqslant 1)$, then $L_3 \subset L_5 \subset \cdots \subset L_{2k+1} \subset L_{2k+3}$ and

$$x \oplus^{2k+3} y = \begin{cases} x \oplus^{2k+1} y, & x+y \leqslant 2k, \\ 2k+1, & x+y = 2k+1, \\ 2k+2, & x+y \geqslant 2k+2. \end{cases}$$

If $n = 2k (k \geqslant 2)$, then $L_4 \subset L_6 \subset \cdots \subset L_{2k} \subset L_{2k+2}$ and

$$x \oplus^{2k+2} y = \begin{cases} x \oplus^{2k} y, & x+y \leqslant 2k-1, \\ 2k, & x+y = 2k, \\ 2k+1, & x+y \geqslant 2k+1, \end{cases}$$

where $x \oplus^{2k+1} y = \min(2k, x+y)$ for instance.

(iv) If $\mathscr{L}_n$ is the canonical LM_n algebra, then the chain $\mathscr{R} = \{r_0 < r_1 < r_2 < \cdots < r_{n-1} < r_n\}$ of functions $r_i : L_n \longrightarrow \{0,1\}$, $i = \overline{0,n}$, where $r_0 x = 0$ and $r_n x = 1$

for any $x \in L_n$, can be organized as an LM_{n+1} algebra isomorphic to the canonical LM_{n+1} algebra.

(v) If $n = 2k+1$, then the operation $\oplus^n$ is expressed in terms of the operations $\vee$, $\wedge$, $r_{k+1}, \dots, r_{2k}$ and the *successor*, while the dual operation, $\cdot^n$, is expressed in terms of the dual operations $\wedge$, $\vee$, $r_1, \dots, r_k$ and the *predecessor*; if $n = 2k$, then the operation $\oplus^n$ is expressed in terms of the operations $\vee$, $\wedge$, $r_k, \dots, r_{2k-1}$ and the *successor*, while the dual operation, $\cdot^n$, is expressed in terms of the dual operations $\wedge, \vee, r_1, \dots, r_k$ and the *predecessor*.

Remark 6.3. Summarizing, we have:

1. The set L_n $(n \geqslant 2)$ can be organized as the canonical MV_n algebra as well as the canonical LM_n algebra. The canonical MV_n algebra, $\mathscr{L}_n^{(\mathrm{MV}_n)}$, can be identified with the canonical LM_n algebra, $\mathscr{L}_n^{(\mathrm{LM}_n)}$ (by the transformations Φ and Ψ).
2. Some subsets of L_n are MV_n subalgebras and/or LM_n subalgebras of the canonical algebra $\mathscr{L}_n$:
   - For $n \in \{2,3,4\}$, MV_n subalgebras of the canonical algebra $\mathscr{L}_n^{(\mathrm{MV}_n)}$ coincide with LM_n subalgebras of the canonical algebra $\mathscr{L}_n^{(\mathrm{LM}_n)}$.
   - For $n \geqslant 5$:
     — Any MV_n subalgebra of the canonical algebra $\mathscr{L}_n^{(\mathrm{MV}_n)}$ is an LM_n subalgebra of the canonical algebra $\mathscr{L}_n^{(\mathrm{LM}_n)}$.
     — Not any LM_n subalgebra of the canonical algebra $\mathscr{L}_n = \mathscr{L}_n^{(\mathrm{LM}_n)}$ is an MV_n subalgebra of the canonical algebra $\mathscr{L}_n = \mathscr{L}_n^{(\mathrm{MV}_n)}$. Indeed, let $\mathscr{L}_n$ be the canonical LM_n algebra. If $n = 2k$ $(k \geqslant 3)$, then $L_{2k} = \{0, 1, \dots, k-1, k, \dots, 2k-2, 2k-1\}$. Then $S'_{2k} = L_{2k} - \{k-1, k\}$ is an LM_{2k} subalgebra of $\mathscr{L}_{2k}$, but it is not an MV_{2k} subalgebra of the canonical MV_{2k} algebra, $\Psi(\mathscr{L}_{2k})$, since it is not closed under $\oplus$: $(k-2) \oplus 1 = k-1 \notin S'_{2k}$. If $n = 2k+1$ $(k \geqslant 2)$, then $L_{2k+1} = \{0, 1, \dots, k-1, k, k+1, \dots, 2k-1, 2k\}$. Then, $S''_{2k+1} = L_{2k+1} - \{k\}$ is an LM_{2k+1} subalgebra of $\mathscr{L}_{2k+1}$, but it is not an MV_{2k+1} subalgebra of the canonical MV_{2k+1} algebra, $\Psi(\mathscr{L}_{2k+1})$, since it is not closed under $\oplus$: $(k-1) \oplus 1 = k \notin S''_{2k+1}$.

Example. For $n = 5$, $L_5 = \{0,1,2,3,4\}$. The MV_5 subalgebras of $\mathcal{L}_5^{(\mathrm{MV}_5)}$ are $\{0,1,2,3,4\}, \{0,2,4\}$, $\{0,4\}$, while the LM_5 subalgebras of $\mathscr{L}_5^{(\mathrm{LM}_5)}$ are $\{0,1,2,3,4\}, \{0,2,4\}$, $\{0,4\}$ and $\{0,1,3,4\}$.

3. By [11, 1.12, 2.8] and the corresponding converse statements:
   - For $n \in \{2,3,4\}$, MV_n algebras coincide with LM_n algebras (by the transformations Φ and Ψ).
   - For $n \geqslant 5$:
     — Any MV_n algebra is an LM_n algebra.

We shall analyse, in the third part of the study, those LM_n algebras which are MV_n algebras (i.e. for which the transformation Ψ is defined), for every $n \geqslant 5$.

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